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LINKING SYSTEMS II

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Linking systems II

by

A. Schrijver

ABSTRACT

This report is a sequel to the report "Linking systems" [6]. Some further results on linking systems are presented. The connections between linking systems and Hocquenghem's tabloid are elucidated. Furthermore a proof is given of theorem 7.5 of [6]: each base of a gammoid is a gammoid-base.

KEY WORDS & PHRASES: *Matroids (gammoids), graphs (bipartite graphs, digraphs), linking systems.*

0. INTRODUCTION

This report is a sequel to [6], where the notion of a linking system was introduced. Hence we use the same notations and definitions and suppose the theorems proven there to be known. In this report we give some further results on linking systems.

In section 1 we give a slightly modified definition of a linking system and prove that this definition is equivalent to the original one in [6]. In [4] HOCQUENGHEM defines the notion of a "tabloid", which is closely related to the concept of linking systems; in fact each linking system is a tabloid. HOCQUENGHEM (private communication) gave a necessary and sufficient condition for a tabloid to be a linking system. For more details see also section 1.

In chapter 2 we modify a theorem of GREENE [3] to a theorem on linking systems.

In chapter 3 we extend a theorem of BRUALDI [1] (on maximal "double-independent" matchings in a bipartite graph with on both sides a matroid) to a theorem on linking systems.

Finally, in section 4 we prove a theorem stated in [6] without full proof, namely theorem 7.5: each base of a gammoid is a gammoid-base. For this we make use of results of INGLETON & PIFF [5].

1. LINKING SYSTEMS AND TABLOIDS

In this section we give an alternative definition of a linking system and show the relation with HOCQUENGHEM's notion of a tabloid.

In [6] we defined a linking system as follows.

<u>DEFINITION</u>. A *linking system* is a triple (X,Y,Λ), such that:

(i) X and Y are finite sets and $\emptyset \neq \Lambda \subset P(X) \times P(Y)$;

(ii) if $X' \overset{\Lambda}{\longleftrightarrow} Y'$, then $|X'| = |Y'|$;

(iii) if $X' \overset{\Lambda}{\longleftrightarrow} Y'$ and $X'' \subset X'$ and $Y'' \subset Y'$, then $X'' \overset{\Lambda}{\longrightarrow} Y'$ and $X' \overset{\Lambda}{\longleftarrow} Y''$;

(iv) if $X'' \overset{\Lambda}{\longleftrightarrow} Y''$ (maximal in (X',Y')) and $X''' \overset{\Lambda}{\longleftrightarrow} Y'''$ (maximal in (X',Y')), then $X'' \overset{\Lambda}{\longleftrightarrow} Y'''$.

In this definition axiom (iv) can be replaced by an axiom (iv'):

<u>THEOREM 1.1</u>. *In the definition above axiom* (iv) *is equivalent to the axiom* (iv') *if* $X' \overset{\Lambda}{\longleftrightarrow} Y'$ *and* $X'' \overset{\Lambda}{\longleftrightarrow} Y''$, *then* $X''' \overset{\Lambda}{\longleftrightarrow} Y'''$ *for some* X''', Y''' *with:*

$X' \subset X''' \subset X' \cup X''$ *and* $Y'' \subset Y''' \subset Y' \cup Y''$.

<u>PROOF</u>. Let (X,Y,Λ) satisfy (i), (ii), (iii), (iv) and let

$$X' \overset{\Lambda}{\longleftrightarrow} Y' \text{ and } X'' \overset{\Lambda}{\longleftrightarrow} Y''.$$

Extend these two linked pairs to linked pairs, maximal in $(X' \cup X'', Y' \cup Y'')$, i.e. let $X' \subset X_1'$, $Y' \subset Y_1'$ and $X_1' \overset{\Lambda}{\longleftrightarrow} Y_1'$ (maximal in $(X' \cup X'', Y' \cup Y'')$), and $X'' \subset X_1''$, $Y'' \subset Y_1''$ and $X_1'' \overset{\Lambda}{\longleftrightarrow} Y_1''$ (maximal in $(X' \cup X'', Y' \cup Y'')$).
Then by axiom (iv): $X_1' \overset{\Lambda}{\longleftrightarrow} Y_1''$.
Furthermore: $X' \subset X_1' \subset X' \cup X''$ and $Y' \subset Y_1'' \subset Y' \cup Y''$.

Conversely, let (X,Y,Λ) satisfy (i), (ii), (iii), (iv').
Let $X'' \overset{\Lambda}{\longleftrightarrow} Y''$ (maximal in (X',Y')) and $X''' \overset{\Lambda}{\longleftrightarrow} Y'''$ (maximal in (X',Y')).
Then, by axiom (iv'), there are X_0 and Y_0 such that:

$$X_0 \overset{\Lambda}{\longleftrightarrow} Y_0 \text{ and } X'' \subset X_0 \subset X'' \cup X''' \text{ and } Y''' \subset Y_0 \subset Y'' \cup Y'''.$$

We now prove that $X_0 = X''$.
Again by axiom (iv'), since:

$$X_0 \overset{\Lambda}{\longleftrightarrow} Y_0 \text{ and } X'' \overset{\Lambda}{\longleftrightarrow} Y'', \text{ and } X'' \subset X_0,$$

we have:

$$X_0 \overset{\Lambda}{\longleftrightarrow} Y_0' \text{ for some } Y_0' \text{ such that } Y'' \subset Y_0' \subset Y_0 \cup Y''.$$

But also:

$$X'' \subset X_0 \subset X'' \cup X''' \subset X',$$

and:

$$Y'' \subset Y_0' \subset Y_0 \cup Y'' \subset Y'' \cup Y''' \subset Y'.$$

Hence, since:

$$X'' \overset{\Lambda}{\longleftrightarrow} Y'' \text{ (maximal in } (X',Y')),$$

it follows that $X'' = X_0$.
By the same method we can prove that $Y''' = Y_0$.
This shows that $X'' \overset{\Lambda}{\longleftrightarrow} Y'''$. □

Next we show a relation between linking systems and tabloids. HOCQUENGHEM [4] defines a tabloid as follows:

DEFINITION. A *tabloid* is a triplet (X,Y,λ) such that:

(i) X and Y are finite sets and $\lambda\colon P(X) \times P(Y) \longrightarrow \mathbb{Z}$;

(ii) for each $X' \subset X$ the function $\lambda_{X'}\colon P(Y) \longrightarrow \mathbb{Z}$, with $\lambda_{X'}(Y') = \lambda(X',Y')$, is a rank-function of a matroid on Y;

(iii) for each $Y' \subset Y$ the function $\lambda_{Y'}\colon P(X) \longrightarrow \mathbb{Z}$, with $\lambda_{Y'}(X') = \lambda(X',Y')$, is a rank-function of a matroid on X.

The notions of linking system and tabloid are interrelated but not the same, as shows the following proposition.

PROPOSITION 1.2. *If* (X,Y,λ) *is a linking system with linking function* λ, *then* (X,Y,λ) *is a tabloid. There is a tabloid* (X,Y,λ) *which is not a linking system.*

PROOF. By the definition of a linking function λ in [6], λ satisfies the axioms in the definition of a tabloid.
For an example that the converse does not hold, let $X = Y = \{a,b\}$, and define $\lambda\colon P(X) \times P(Y) \longrightarrow \mathbb{Z}$ by:

$$\lambda(X',Y') = \min.\{1, |X' \cap Y'|\}.$$

Then (X,Y,λ) is a tabloid but not a linking system. □

HOCQUENGHEM (private communication) showed furthermore:

THEOREM 1.3. (HOCQUENGHEM) *Let* (X,Y,λ) *be a tabloid. Then the following conditions are equivalent:*

(1) *for each* $X' \subset X$, $Y' \subset Y$, $x \in X\setminus X'$, $y \in Y\setminus Y'$ *it is true that:*
$\lambda(X,Y \cup \{y\}) + \lambda(X \cup \{x\},Y) \leq \lambda(X,Y) + \lambda(X \cup \{x\},Y \cup \{y\})$;

(2) (X,Y,λ) *is a linking system.*

PROOF. (2) $\Longrightarrow$ (1): Straightforward from the definition of a linking system;
(1) $\Longrightarrow$ (2): We have only to prove: if $X', X'' \subset X$ and $Y', Y'' \subset Y$ then:

$$\lambda(X'\cup X'',Y'\cap Y'') + \lambda(X'\cap X'',Y'\cup Y'')\leq \lambda(X',Y') + \lambda(X'',Y'').$$

Let $A \subset X$, $B \subset Y$, $\{a_1,\dots,a_m\} \subset X\setminus A$, $\{b_1,\dots,b_n\} \subset Y\setminus B$. Then:

$$\begin{array}{llll}
\lambda(A,B\cup\{b_1\}) & -\lambda(A,B) & \leq \lambda(A\cup\{a_1\},B\cup\{b_1\}) & -\lambda(A\cup\{a_1\},B), \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\lambda(A,B\cup\{b_1,\dots,b_n\}) & -\lambda(A,B\cup\{b_1,\dots,b_{n-1}\}) & \leq \lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_n\}) & -\lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_{n-1}\})
\end{array}$$

Hence:

$$\lambda(A,B\cup\{b_1,\dots,b_n\})-\lambda(A,B) \leq \lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_n\})-\lambda(A\cup\{a_1\},B).$$

So:

$$\lambda(A\cup\{a_1\},B)-\lambda(A,B) \leq \lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_n\})-\lambda(A,B\cup\{b_1,\dots,b_n\}).$$

By the same method:

$$\lambda(A\cup\{a_1,\dots,a_m\},B)-\lambda(A,B) \leq \lambda(A\cup\{a_1,\dots,a_m\},B\cup\{b_1,\dots,b_n\})-\lambda(A,B\cup\{b_1,\dots,b_n\})$$

or:

$$\lambda(A\cup\{a_1,\dots,a_m\},B)+\lambda(A,B\cup\{b_1,\dots,b_n\}) \leq \lambda(A,B)+\lambda(A\cup\{a_1,\dots,a_m\},B\cup\{b_1,\dots,b_n\}) \quad \dots(*)$$

Now we have:

$\lambda(X'\cap X'',Y'\cup Y'') + \lambda(X'\cup X'',Y'\cap Y'') \leq$ (since (X,Y,λ) is a tabloid)

$\lambda(X'\cap X'',Y') + \lambda(X'\cap X'',Y'') - \lambda(X'\cap X'',Y'\cap Y'') +$

$\lambda(X',Y'\cap Y'') + \lambda(X'',Y'\cap Y'') - \lambda(X'\cap X'',Y'\cap Y'') \leq$ (using (*))

$\lambda(X',Y') + \lambda(X'',Y'')$. □

2. A THEOREM ON LINKED PAIRS

In this section we use the theorem of EDMONDS [2] on the union of matroids to get a theorem on linking systems. The theorem was inspired by a result of GREENE (see GREENE [3] or WOODALL [8]).

THEOREM 2.1. *Let* (X,Y,Λ) *be a linking system and* $(X',Y') \in \Lambda$. *Furthermore, let* $X'' \subset X'$. *Then:* $X'' \overset{\Lambda}{\longleftrightarrow} Y''$ *and* $(X'\backslash X'') \overset{\Lambda}{\longleftrightarrow} (Y'\backslash Y'')$ *for some* $Y'' \subset Y'$.

PROOF. Let $M_1 = (Y',I_1)$ be the matroid on Y' with:

$$I_1 = \{Y'_0 \subset Y' \mid X'' \overset{\Lambda}{\longleftarrow} Y'_0\}.$$

Let $M_2 = (Y',I_2)$ be the matroid on Y' with:

$$I_2 = \{Y'_0 \subset Y' \mid (X'\backslash X'') \overset{\Lambda}{\longleftarrow} Y'_0\}.$$

If we have that Y' is a base of $M_1 \vee M_2$, then there exists a $Y'' \subset Y'$ such that:

$$X'' \overset{\Lambda}{\longleftrightarrow} Y'' \text{ and } (X'\backslash X'') \overset{\Lambda}{\longleftrightarrow} (Y'\backslash Y'').$$

EDMONDS' theorem says: $(\rho_1 \vee \rho_2)(Y') = |Y'|$ iff for each $Y'_0 \subset Y'$:

$$\rho_1(Y'_0) + \rho_2(Y'_0) + |Y'\backslash Y'_0| \geq |Y'|, \text{ i.e.}$$

for each $Y'_0 \subset Y'$: $\rho_1(Y'_0) + \rho_2(Y'_0) \geq |Y'_0|$.
We will prove that this latter inequality always holds. Let $Y'_0 \subset Y'$. Then $X'_0 \overset{\Lambda}{\longleftrightarrow} Y'_0$ for some $X'_0 \subset X'$.

Then it follows:

$$(X_0' \cap X'') \xrightarrow{\Lambda} Y_0', \text{ i.e. } \rho_1(Y_0') \geq |X_0' \cap X''|;$$

also:

$$(X_0' \backslash X'') \xrightarrow{\Lambda} Y_0', \text{ i.e. } \rho_2(Y_0') \geq |X_0' \backslash X''|.$$

Hence:

$$\rho_1(Y_0') + \rho_2(Y_0') \geq |X_0' \cap X''| + |X_0' \backslash X''| = |X_0'| = |Y_0'|.$$

As this is true for each $Y_0' \subset Y'$, we have shown that

$$(\rho_1 \vee \rho_2)(Y') = |Y'|, \text{ i.e. } Y' \text{ is a base of } M_1 \vee M_2. \qquad \square$$

3. INDEPENDENT LINKED PAIRS

In this section we generalize a theorem of BRUALDI [1] (see corollary 3.2.). This generalization is:

THEOREM 3.1. *Let* (X,Y,Λ) *be a linking system and let* $(X,\mathcal{I})$ *and* $(Y,\mathcal{J})$ *be matroids. Then:* $\max\{|Y'| \mid \text{there is an } X' \in \mathcal{I} \text{ such that } X' \xleftrightarrow{\Lambda} Y' \in \mathcal{J}\} =$

$$\min_{\substack{X' \subset X \\ Y' \subset Y}} (\rho(X') + \sigma(Y') + \lambda(X\backslash X', Y\backslash Y')),$$

where ρ *is the rank-function of* $(X,\mathcal{I})$ *and* σ *the rank-function of* $(Y,\mathcal{J})$.

PROOF. As is done by WELSH [7] we use EDMONDS' intersection theorem [2]: if $(Y,\mathcal{J}_1)$ and $(Y,\mathcal{J}_2)$ are matroids, with rank-functions ρ_1 and ρ_2, respectively, then $\max\{|Y'| \mid Y' \in \mathcal{J}_1 \text{ and } Y' \in \mathcal{J}_2\} = \min_{Y' \subset Y} (\rho_1(Y') + \rho_2(Y\backslash Y'))$.
We have:

$\max\{|Y'| \mid I \ni X' \overset{\Lambda}{\longleftrightarrow} Y' \in J\} =$

$\max\{|Y'| \mid Y' \in J \text{ and } Y' \in I*\Lambda\} =$

$\min_{Y' \subset Y}(\sigma(Y') + (\rho*\lambda)(Y\backslash Y')) =$ [following thm. 4.1 in [6]]

$\min_{Y' \subset Y}(\sigma(Y') + \min_{X' \subset X}(\rho(X') + \lambda(X\backslash X', Y\backslash Y'))) =$

$\min_{\substack{X' \subset X \\ Y' \subset Y}}(\rho(X') + \sigma(Y') + \lambda(X\backslash X', Y\backslash Y')).$ □

BRUALDI's theorem follows from this. A more straightforward proof of that theorem can be found in WELSH [7].

<u>COROLLARY 3.2</u>. (BRUALDI [1]). *Let* (X,Y,E) *be a bipartite graph and let* (X,I) *and* (Y,J) *be matroids (with rank-function* ρ *and* σ, *respectively). Then:*

$\max\{|Y'| \mid Y' \in J \text{ and } Y' \text{ is matched in E with an } X' \in I\} =$

$\min_{X' \subset X}(\sigma(E(X')) + \rho(X\backslash X')).$

<u>PROOF</u>. The bipartite graph (X,Y,E) generates a linking system (X,Y,δ_E). In this linking system δ_E is given by $\delta_E(X',Y') = \max\{|X''| \mid X'' \subset X'$ is matched in E with some $Y'' \subset Y'\} = \min\{|X''| + |Y''| \mid X'' \subset X', Y'' \subset Y',$ (X'',Y'') is covering for $(X',Y',E')\}$ (see [6], section 2).

Hence, by theorem 3.1.:

$\max\{|Y'| \mid Y' \in J \text{ and } Y' \text{ is matched in E with an } X' \in I\} =$

$\min_{\substack{X' \subset X \\ Y' \subset Y}}(\rho(X') + \sigma(Y') + \delta_E(X\backslash X', Y\backslash Y')) =$

$\min\{(\rho(X') + \sigma(Y') + |X''| + |Y''|) \mid X' \subset X, Y' \subset Y, X'' \subset X\backslash X', Y'' \subset Y\backslash Y',$ $(X'',Y'') \text{ is a } (X\backslash X', Y\backslash Y') \text{ covering}\} =$

$\min\{(\rho(X') + \sigma(Y') + |X''| + |Y''|) \mid X' \subset X, Y' \subset Y, X'' \subset X\backslash X',$ $E(X\backslash(X' \cup X''))\backslash Y' \subset Y'' \subset Y\backslash Y'\} =$

$\min\{(\rho(X') + \sigma(Y') + |X''| + |E(X\backslash(X' \cup X''))\backslash Y'|) \mid X' \subset X, Y' \subset Y, X'' \subset X\backslash X'\} =$

$\min_{\substack{X' \subset X \\ Y' \subset Y}}(\rho(X') + \sigma(Y') + |E(X\backslash X')\backslash Y'|) =$

$\min_{X' \subset X}(\rho(X') + \sigma(E(X\backslash X'))).$ □

4. GAMMOIDS AND GAMMOID LINKING SYSTEMS

In this section we give a proof of some propositions on gammoids and gammoid linking systems, already stated without proof in [6]. In this proof we use the methods of INGLETON & PIFF [5]. First we give the definition of a gammoid.

DEFINITION. Let (Z,Γ) be a directed graph and let $X,Y \subset Z$. Let I be the set $\{Y' \subset Y \mid$ there are $|Y'|$ pairwise vertex-disjoint paths starting in X and ending in $Y'\}$. Then the so-obtained matroid (Y,I) is called a *gammoid*. If $Y = Z$ it is called a *strict gammoid*.

Clearly, a matroid M is a gammoid if and only if M is a restriction of a strict gammoid.

Next we mention the theorem of INGLETON & PIFF [5], which shows that every transversal matroid is the dual of a strict gammoid, and vice versa.

THEOREM 4.1. (INGLETON & PIFF [5]). *Let* (X,Y,E) *be a bipartite graph such that there is a linking* $\ell\colon X \longrightarrow Y$ *in* E (*see* [6], *section* 1). *Embed* X *in* Y *by means of the function* ℓ *and define the digraph* (Y,Γ) *by:*

$$\Gamma = \{(y,x) \in Y \times X \mid (x,y) \in E\}.$$

Let M *be the matroid* (Y,I), *where:*

$$I = \{Y' \subset Y \mid Y' \text{ is matched in } E \text{ with some subset of } X\}.$$

Let N *be the matroid* (Y,J), *where:*

$$J = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths in } \Gamma, \text{ starting in } Y\backslash X \text{ and ending in } Y'\}.$$

Then: $M^* = N$.

PROOF. See [5]. $\square$

We use this nice result to prove the next theorem. In [6] we have defined a gammoid as follows.
Let (Z,Γ) be a directed graph and X and Y disjoint subsets of Z. Let furthermore:

$$\mathcal{B} = \{(X\backslash X') \cup Y' \mid X' \text{ and } Y' \text{ are linked in } \Gamma\}.$$

Then we called $(X\cup Y,\mathcal{B})$ a gammoid and X a gammoid-base of the gammoid. (We recall that $\mathcal{B}$ is used to refer to the set of bases of the matroid.) From the following theorem it follows that this last definition of a gammoid is equivalent to the definition at the beginning of this section. Also it follows that each base of a gammoid is a gammoid-base (cf. [6], theorem 7.5).

THEOREM 4.2. *Let* $G = (Y,\mathcal{I})$ *be a gammoid (as defined in the second paragraph of this section) and let* B *be a base of* G. *Then there exists a directed graph* (Z,Γ) *such that* $Y \subset Z$ *and such that the set of bases of* G *is*

$$\mathcal{B} = \{Y' \subset Y \mid B \;\&\; Y' \text{ are linked in } \Gamma\}.$$

PROOF. Let $G = (Y,\mathcal{I})$ be a gammoid and let B be a base of G. Then there exists a digraph (Z',Γ') and an $X' \subset Z'$ such that $Y \subset Z'$ and:

$$\mathcal{I} = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths in } \Gamma' \text{ starting in } X' \text{ and ending in } Y'\}.$$

Let $(Z'\backslash X',Z',E)$ be the bipartite graph with:

$$(z_1,z_2) \in E \text{ iff } z_1 = z_2 \text{ or } (z_2,z_1) \in \Gamma'.$$

Let M be the matroid $(Z',\mathcal{J})$ where:

$$\mathcal{J} = \{Z'' \subset Z' \mid \text{there is a } W \subset Z'\backslash X' \text{ such that } W \;\&\; Z'' \text{ are linked in } E\}.$$

Let $M^* = (Z', \mathcal{J}^*)$ be the dual matroid of M.
By theorem 4.1. we know:

$$\mathcal{J}^* = \{Z'' \subset Z' \mid \text{there are } |Z''| \text{ pairwise vertex-disjoint paths in } \Gamma' \text{ starting in } X' \text{ and ending in } Z''\}.$$

Hence the restriction of M^* to $Y \subset Z'$ is

$$M^* \times Y = (Y, \mathcal{J}^* \times Y) = (Y, \mathcal{I}).$$

So:

$$G = (Y, \mathcal{I}) = M^* \times Y = (M \cdot Y)^*.$$

Let V be a base of $M \times (Z' \backslash Y)$. Then, as is well-known:

$$M \cdot Y = (M \times (V \cup Y)) \cdot Y.$$

Now $M \times (V \cup Y)$ is the matroid $(V \cup Y, \mathcal{J} \times (V \cup Y))$, where:

$$\mathcal{J} \times (V \cup Y) = \{U \subset V \cup Y \mid U \text{ is matched in } E \text{ with some subset of } Z' \backslash X'\}.$$

Also:

$$Y' \in (\mathcal{J} \times (V \cup Y)) \cdot Y \text{ iff } Y' \cup V \in \mathcal{J} \times (V \cup Y),$$

i.e. iff $Y' \cup V$ is matched in E with some subset of $Z' \backslash X'$.
Since B is a base of $G = (M \cdot Y)^*$, we have:

$$Z' \backslash X' \ \& \ (Y \backslash B) \cup V \text{ are linked in } E.$$

Let $\ell: Z' \backslash X' \longrightarrow (Y \backslash B) \cup V$ be a linking. Now let $Z = Y \cup V$ and embed $Z' \backslash X'$ in Z by means of the linking ℓ.

Furthermore let $\Gamma = \{(z_1,z_2) \in Z \times Z \mid (z_2,z_1) \in E\}$.
Then, again by theorem 4.1, we know that:

$(J\times(V\cup Y))^* = \{Y' \subset Z \mid$ there are $|Y'|$ pairwise vertex-disjoint paths in Γ starting in B and ending in $Y'\}$.

Now:

$I = ((J\times(V\cup Y))\cdot Y)^* = (J\times(V\cup Y))^* \times Y = \{Y' \subset Y \mid$ there are $|Y'|$ pairwise vertex-disjoint paths in Γ starting in B and ending in $Y'\}$. □

As a corollary we have:

COROLLARY 4.3. *Let* $G = (X\cup Y,\mathcal{B})$ *be a gammoid,* $X \cap Y = \emptyset$ *and* X *a base of* G. *Then there exists a digraph* (Z,Γ) *such that:*
$X \cup Y \subset Z$ *and* $\mathcal{B} = \{(X\backslash X') \cup Y' \mid X'$ & Y' *are linked in* $\Gamma\}$.

PROOF. Construct the digraph (Z,Γ) of theorem 4.2, taking $Y := X \cup Y$ and $B := X$. Then $X \cup Y \subset Z$ and $\mathcal{B} = \{W \subset X \cup Y \mid X$ & W are linked in $\Gamma\}$.
Also we have:
X' & Y' are linked in Γ iff X & $(X\backslash X') \cup Y'$ are linked in Γ, i.e. iff $(X\backslash X') \cup Y' \in \mathcal{B}$.
Hence:

$\mathcal{B} = \{(X\backslash X') \cup Y' \mid X'$ & Y' are linked in $\Gamma\}$. □

REFERENCES

[1] BRUALDI, R.A., *Admissible mappings between dependence spaces*, Proc. London Math. Soc. (3) 21 (1970) 296-312.

[2] EDMONDS, J., *Submodular functions, matroids, and certain polyhedra*, Lectures, Calgary International Symposium on Combinatorial Structures, June 1969.

[3] GREENE, C., *A multiple exchange property for bases*, Proc. Amer. Math. Soc., 39 (1973), 45-50.

4 HOCQUENGHEM, S., *Tabloïdes*, Communication International Congress of Mathematicians, Vancouver, 1974.

5 INGLETON, A.W. & M.J. PIFF, *Gammoids and Transversal Matroids*, J. Combinatorial Theory (B) 15 (1973) 51-68.

6 SCHRIJVER, A., *Linking systems*, Report ZW 29/74 Afdeling Zuivere Wiskunde Mathematisch Centrum, Amsterdam, 1974.

7 WELSH, D.J.A., *On matroid theorems of Edmonds and Rado*, J. London Math. Soc. (2) 2 (1970) 251-256.

8 WOODALL, D.R., *An Exchange Theorem for Bases of Matroids*, J. Combinatorial Theory (B) 16 (1974) 227-228.